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Stochastic Equations of the Langevin Type under a Weakly Dependent Perturbation

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Asymptotic expansions for the probability density of the solution of a stochastic differential equation under a weakly dependent perturbation are proposed. In particular, linear partial differential equations for the first two terms of the correlation time expansion are derived. It is shown that in these expansions the boundary layer part appears and non-Gaussianity of the perturbation is important for the Fokker–Planck approximation correction.

KEY WORDS: Langevin equation; stochastic Liouville equation; Fokker–Planck equation; correlation time expansion.

1. INTRODUCTION

The description of statistical systems by means of stochastic differential equations of the Langevin type is a powerful method of their analysis.[1–3] As a matter of convenience it is often assumed that the stochastic perturbation in these equations is white noise. In this case the solution is Markovian and the Fokker–Planck equation exists for the probability density. If the stochastic perturbation is not white noise, then the solution of the stochastic differential equation is a non-Markovian process and in general one cannot obtain a closed equation for the probability density.

We consider the N-dimensional equation of the Langevin type

$$\dot{x}_j(t) = a_j(t, x, \varepsilon) + \xi(t\varepsilon^{-2}, x)\, b_j(t, x, \varepsilon), \qquad t > 0, \quad j = 1,..., N \tag{1}$$

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where ε is a small positive parameter, and the nonrandom functions $a_j(t, x, \varepsilon)$, $b_j(t, x, \varepsilon)$ have the asymptotic expansions

$$a_j(t, x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k a_{jk}(t, x), \qquad b_j(t, x, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k b_{jk}(t, x) \tag{2}$$

The zero-mean random function $\xi_\varepsilon(t, x) \equiv \xi(t\varepsilon^{-2}, x)$ is strictly stationary in t. It satisfies a weakly dependent condition, namely the cumulants

$$K_m(t - t_1,..., t - t_m, y, y_1,..., y_m) \equiv \langle\langle \xi(t, y)\, \xi(t_1, y_1) \cdots \xi(t_m, y_m)\rangle\rangle$$

of the random function $\xi(t, x)$ satisfy the inequalities

$$|K_m(t - t_1,..., t - t_m, y, y_1,..., y_m)|$$
$$\leqslant \beta_m \exp\left\{ -\alpha \sum_{k=1}^{m} |t - t_k| \right\}, \qquad \beta_m, \alpha > 0 \tag{3}$$

It follows from this and from the properties of the cumulants[1, 4, 5] that the values of function $\xi_\varepsilon(t, x)$ are almost independent for different t and small ε.

Let $\rho(t, x)$ be the probability density for the N-component solution of Eq. (1). It satisfies the stochastic Liouville equation[1, 4]

$$\frac{\partial \rho(t, x)}{\partial t} = -\frac{\partial}{\partial x_j} [a_j(t, x, \varepsilon)\, \rho(t, x)] - \frac{\partial}{\partial x_j} [\xi_\varepsilon(t, x)\, b_j(t, x, \varepsilon)\, \rho(t, x)] \tag{4}$$

Then, according to van Kampen's lemma,[1] the probability density $P(t, x)$ for the N-component process $x(t)$ is the average $P(t, x) = \langle \rho(t, x)\rangle$ over the distribution of $\xi_\varepsilon(t, x)$. One can represent the formal solution of Eq. (4) by means of the time-ordered exponential and obtain[4]

$$P(t, x) = \left\langle \underset{\leftarrow}{T} \exp\left\{ -\int_0^t F(s, x)\, ds \right\} \right\rangle \rho(0, x) \tag{5}$$

where $\underset{\leftarrow}{T}\exp$ denotes the time-ordered exponential, and $F(s, x)$ denotes the differential operator

$$F(s, x) \equiv \frac{\partial}{\partial x_j} a_j(t, x, \varepsilon) + \frac{\partial}{\partial x_j} \xi_\varepsilon(t, x)\, b_j(t, x, \varepsilon)$$

It is difficult to evaluate the right-hand side of formula (5). The method of ordered operator cumulants[1, 4–7] is the most general approach developed for this purpose. In conformity with it, some approximations for finding $P(t, x)$ were obtained in refs. 8 and 9. Another approach consists in the construction of the so-called best Fokker–Planck approximations.[1, 10–12] For this purpose the projection approach[13] and the functional-calculus approach[14] also have been applied.

All of these methods were applied to the Gaussian model of the random perturbation in Eq. (1) and for $a_j(t, x, \varepsilon) = a_j(x)$, $b_j(t, x, \varepsilon) = \varepsilon^{-1} b_j(x)$, and $\xi_\varepsilon(t, x) = \xi_\varepsilon(t)$.

The major aim of this paper is to propose an approach to the general non-Gaussian model of noise in (1). It consists in the reduction of the problem of computation of the right-hand side of (5) to the construction of the solution of an infinite chain of integrodifferential equations containing a small parameter in derivatives. For this chain we use an asymptotic method for singular perturbed equations (ref. 15, Chapter 4; ref. 16) and it enables us to obtain for $P(t, x)$ the asymptotic formula

$$P(t, x) = \sum_{k=0}^{\infty} \varepsilon^k (g_k(t, x) + z_k(\tau, x)) \tag{6}$$

where $\tau = t\varepsilon^{-2}$ is a stretched variable.

This approach makes it possible to find successively closed equations for $g_0(t, x)$, $z_0(\tau, x)$, $g_1(t, x)$, $z_1(\tau, x)$,.... If $a_j(t, x, \varepsilon) = a_j(t, x)$, $b_j(t, x, \varepsilon) = \varepsilon^{-1} b_j(t, x)$, formula (6) gives the correlation time expansion. Moreover, $g_0(t, x)$ is the solution of the Fokker–Planck equation and other terms of (6) can be obtained through the solution of the Fokker–Planck equation by means of quadrature formulas. The so-called boundary layer terms $z_0(\tau, x)$, $z_1(\tau, x)$,..., are of vital importance for t near $t = 0$.

Note that in the Gaussian case, $g_1(t, x) \equiv 0$, and so the non-Gaussianity of the perturbation in Eq. (1) is important for the correction of the Fokker–Planck approximation.

2. A SINGULAR PERTURBED INFINITE CHAIN OF INTEGRODIFFERENTIAL EQUATIONS FOR $P(t, x)$

When we take the average of Eq. (1) over the distribution of $\xi_\varepsilon(t, x)$ we find the average $\langle \xi_\varepsilon(t, x) \, \rho(t, x) \rangle$. In general one cannot express it through $P(t, x)$ and obtain a closed equation for $P(t, x)$. We use the following formula[17, 18] that allows us to find this average through the average of variational derivatives of $\rho(t, x)$:

$$\langle \xi_\varepsilon(t, x)\, \rho(t, x) \rangle$$
$$= \sum_{m=1}^{\infty} \frac{1}{m!} \int_0^t \cdots \int_0^t \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K_m(\varepsilon^{-2}(t-s_1),..., \varepsilon^{-2}(t-s_m), x, y_1,..., y_m)$$
$$\times \left\langle \frac{\delta^m \rho(t, x)}{\delta\xi_\varepsilon(s_1, y_1) \cdots \delta\xi_\varepsilon(s_m, y_m)} \right\rangle dy_1 \cdots dy_m\, ds_1 \cdots ds_m \tag{7}$$

Writing Eq. (4) in the integral form of t, we have for the variational derivatives the equations

$$\frac{\delta^m \rho(t, x)}{\delta\xi_\varepsilon(s_1, y_1) \cdots \delta\xi_\varepsilon(s_m, y_m)}$$
$$= -\int_0^t \frac{\partial}{\partial x_j} \left\{ [a_j(t_1, x, \varepsilon) - \xi_\varepsilon(t_1, x)\, b_j(t_1, x, \varepsilon)] \right.$$
$$\left. \times \frac{\delta^m \rho(t_1, x)}{\delta\xi_\varepsilon(s_1, y_1) \cdots \delta\xi_\varepsilon(s_m, y_m)} \right\} dt_1$$
$$- \sum_{n=1}^{m} \theta(t-s_n) \frac{\partial \delta(x-y_n)}{\partial x_j} b_j(s_n, y_n, \varepsilon)$$
$$\times \frac{\delta^{m-1} \rho(s_n, y_n)}{\delta\xi_\varepsilon(s_1, y_1) \cdots \delta\xi_\varepsilon(s_{n-1}, y_{n-1})\, \delta\xi_\varepsilon(s_{n-1}, y_{n+1}) \cdots \delta\xi_\varepsilon(s_m, y_m)} \tag{8}$$

where $\theta(t)$ is the Heaviside unit function and $\delta(x)$ is the Dirac delta function.

Note that formula (7) has sense when instead of $\rho(t, x)$ we substitute the variational derivatives of $\rho(t, x)$. Taking it, (4), and (8) into account, we obtain for $P(t, x) = \langle \rho(t, x) \rangle$ the infinite chain of integrodifferential equations

$$\frac{\partial \langle \rho(t, x) \rangle}{\partial t}$$
$$= \frac{\partial}{\partial x_j} [a_j(t, x, \varepsilon) \langle \rho(t, x) \rangle]$$
$$- \frac{\partial}{\partial x_j} \left[b_j(t, x, \varepsilon) \sum_{m=1}^{\infty} \frac{1}{m!} \right.$$
$$\times \int_0^t \cdots \int_0^t \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K_m \left(\frac{t-s_1}{\varepsilon^2},..., \frac{t-s_m}{\varepsilon^2}, x, y_1,..., y_m \right)$$
$$\left. \times \left\langle \frac{\delta^m \rho(t, x)}{\delta\xi_\varepsilon(s_1, y_1) \cdots \delta\xi_\varepsilon(s_m, y_m)} \right\rangle dy_1 \cdots dy_m\, ds_1 \cdots ds_m \right]$$

$$\left\langle \frac{\delta^m \rho(t,x)}{\delta\xi_\varepsilon(s_1,y_1)\cdots d\xi_\varepsilon(s_m,y_m)} \right\rangle$$

$$= -\int_0^t \frac{\partial}{\partial x_j}\left[a_j(t_1,x,\varepsilon)\left\langle \frac{\delta^m \rho(t_1,x)}{\delta\xi_\varepsilon(s_1,y_1)\cdots \delta\xi_\varepsilon(s_m,y_m)}\right\rangle\right] dt_1$$

$$-\int_0^t \frac{\partial}{\partial x_j}\left[b_j(t_1,x,\varepsilon)\sum_{k=1}^{\infty}\frac{1}{k!}\right.$$

$$\times \int_0^{t_1}\cdots\int_0^{t_1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K_k\left(\frac{t_1-s_{m+1}}{\varepsilon^2},\dots,\frac{t_1-s_{m+k}}{\varepsilon^2},x,y_{m+1},\dots,y_{m+k}\right)$$

$$\times\left\langle \frac{\delta^{m+k}\rho(t_1,x)}{\delta\xi_\varepsilon(s_1,y_1)\cdots\delta\xi_\varepsilon(s_{m+k},y_{m+k})}\right\rangle$$

$$\left.\times\, dy_{m+1}\cdots dy_{m+k}\, ds_{m+1}\cdots ds_{m+k}\right] dt_1$$

$$-\sum_{n=1}^{\infty}\theta(t-s_n)\frac{\partial\delta(x-y_n)}{\partial x_j} b_j(s_n,y_n,\varepsilon)$$

$$\times\left\langle \frac{\delta^{m-1}\rho(s_n,y_n)}{\delta\xi_\varepsilon(s_1,y_1)\cdots\delta\xi_\varepsilon(s_{n-1},y_{n-1})\,\delta\xi_\varepsilon(s_{n+1},y_{n+1})\cdots\delta\xi_\varepsilon(s_m,y_m)}\right\rangle,$$

$$m=1,2 \qquad (9)$$

It follows from (3) that the Laplace transforms

$$F_m(x,y_1,p_1,\dots,y_m,p_m)$$

$$=\int_0^\infty\int_0^\infty \exp\left(-\sum_{n=1}^{m} p_n t_n\right) K_m(t_1,\dots,t_m,x,y_1,\dots,y_m)\,dt_1\cdots dt_m$$

exist for $-\alpha<\operatorname{Re} p_n<0$, $n=1,2,3,\dots,m$.

Let us introduce the following notation: $w_0(t,x)=\langle\rho(t,x)\rangle$,

$$w_m(t,x,y_1,p_1,\dots,y_m,p_m)$$

$$=\varepsilon^{-m}\int_0^t\cdots\int_0^t \exp\left[\sum_{n=1}^{m} p_n\varepsilon^{-2}(t-s_n)\right]$$

$$\times\left\langle \frac{\delta^m\rho(t,x)}{\delta\xi_\varepsilon(s_1,y_1)\cdots\delta\xi_\varepsilon(s_m,y_m)}\right\rangle ds_1\cdots ds_m, \qquad m=1,2,3,\dots$$

Then, using the inverse Laplace transform formula, we can reduce the chain (9) to an infinite chain of integrodifferential equations containing a small parameter in the derivatives

$$\frac{\partial w_0(t,x)}{\partial t} = -\frac{\partial}{\partial x_j}[a_j(t,x,\varepsilon)\,w_0(t,x)] - \frac{\partial}{\partial x_j}\Bigg[b_j(t,x,\varepsilon)\sum_{m=1}^{\infty}\frac{\varepsilon^m}{m!\,(2\pi i)^m} \times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int F_m(x,y_1,p_1,\ldots,y_m,p_m) \times w_m(t,x,y_1,p_1,\ldots,y_m,p_m)\,dp_1\cdots dp_m\,dy_1\cdots dy_m\Bigg]$$

$$w_0(0,x)=\rho(0,x)\equiv\delta(x-x(0)) \tag{10}$$

$$\varepsilon^2\frac{\partial w_m(t,x,y_1,p_1,\ldots,y_m,p_m)}{\partial t} = \sum_{n=1}^{m}\Bigg[p_n w_m(t,x,y_1,p_1,\ldots,y_m,p_m) - \varepsilon\frac{\partial\delta(x-y_n)}{\partial x_j}b_j(t,y_n,\varepsilon) \times w_{m-1}(t,y_n,y_1,p_1,\ldots,y_{n-1},p_{n-1},y_{n+1},p_{n+1},\ldots,y_m,p_m)\Bigg]$$
$$-\varepsilon^2\frac{\partial}{\partial x_j}[a_j(t,x,\varepsilon)\,w_m(t,x,y_1,p_1,\ldots,y_m,p_m)]$$
$$-\frac{\partial}{\partial x_j}\Bigg[b_j(t,x,\varepsilon)\sum_{k=1}^{\infty}\frac{\varepsilon^{k+2}}{k!\,(2\pi i)^k} \times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int\cdots\int F_m(x,y_{m+1},p_{m+1},\ldots,y_{m+k},p_{m+k})$$
$$\times w_{m+k}(t,x,y_1,p_1,\ldots,y_{m+k},p_{m+k})\,dp_{m+1}\cdots dp_{m+k} \times dy_{m+1}\cdots dy_{m+k}\Bigg],\quad w_m(0,x,y_1,p_1,\ldots,y_m,p_m)=0,\qquad m=1,2,3,\ldots$$

Here and below the integrals are taken along the straight lines Re $p_k = -\sigma_k < 0$, $k=1,2,3,\ldots$, if integration boundaries are not determined.

Since in the chain (10) Re $p_k<0$, we use the approach from ref. 15, Chapter 4, and ref. 16. Namely, let us seek the solution of this chain in the form

$$w_0(t,x)=\sum_{k=0}^{\infty}\varepsilon^k(w_{0k}(t,x)+z_{0k}(\tau,x))$$
$$w_m(t,x,y_1,p_1,\ldots,y_m,p_m)=\sum_{k=0}^{\infty}\varepsilon^k(w_{mk}(t,x,y_1,p_1,\ldots,y_m,p_m) + z_{mk}(\tau,x,y_1,p_1,\ldots,y_m,p_m)) \tag{11}$$

where $\tau = t\varepsilon^{-2}$, $m = 1, 2, 3,\ldots$, and the functions w_{0k}, z_{0k}, w_{mk}, z_{mk} must satisfy the initial conditions

$$\begin{gathered} w_{00}(0, x) + z_{00}(0, x) = \rho(0, x) \\ w_{0k}(0, x) + z_{0k}(0, x) = 0 \\ w_{mk}(0, x, y_1, p_1,\ldots, y_m, p_m) + z_{mk}(0, x, y_1, p_1,\ldots, y_m, p_m) = 0 \end{gathered} \tag{12}$$

and

$$\begin{gathered} \lim_{\tau \to \infty} z_{0k}(\tau, x) = 0, \qquad \lim_{\tau \to \infty} z_{mk}(\tau, x, y_1, p_1,\ldots, y_m, p_m) = 0 \\ k = 0, 1, 2, 3,\ldots, \qquad m = 1, 2, 3,\ldots \end{gathered} \tag{13}$$

Substituting the regular parts

$$\sum_{k=0}^{\infty} \varepsilon^k w_{0k}(t, x), \qquad \sum_{k=0}^{\infty} \varepsilon^k w_{mk}(t, x, y_1, p_1,\ldots, y_m, p_m)$$

of the expansions (11) instead of $w_0(t, x)$ and $w_m(t, x, y_1, p_1,\ldots, y_m, p_m)$ into (10) and equating coefficients of $\varepsilon^0, \varepsilon^1,\ldots$, one can successively obtain closed integrodifferential equations for functions $w_{0k}(t, x)$, $k = 0, 1,\ldots$, but their solutions do not satisfy in general the initial conditions for the chain (10). The boundary layer parts

$$\sum_{k=0}^{\infty} \varepsilon^k z_{0k}(\tau, x), \qquad \sum_{k=0}^{\infty} \varepsilon^k z_{mk}(\tau, x, y_1, p_1,\ldots, y_m, p_m)$$

serve this purpose.

Introducing the stretched variable $\tau = t\varepsilon^{-2}$ in (10), substituting the boundary layer parts into (10), expanding in Taylor series $a_{jk}(t, x)$ and $b_{jk}(t, x)$ about $t = 0$, and equating coefficients of $\varepsilon^0, \varepsilon^1,\ldots$, one can successively obtain closed equations for $z_{0k}(\tau, x)$, $k = 0, 1,\ldots$.

3. CORRELATION TIME EXPANSION FOR THE PROBABILITY DENSITY $P(t, x)$

Let us suppose that $a_j(t, x, \varepsilon) = a_j(t, x)$, $b_j(t, x, \varepsilon) = \varepsilon^{-1} b_j(t, x)$. In this case the parameter ε defines the correlation time of the random perturbation in Eq. (1). Following the above-mentioned approach, we shall obtain linear partial differential equations for the first two terms of the asymptotic expansion of $P(t, x)$ in powers of ε.

Substituting the regular and boundary layer parts of the expansions (11) in (10) and equating coefficients of ε^0, ε^1, we get the equations for $w_{00}(t, x)$, $w_{01}(t, x)$:

$$\frac{\partial w_{0k}(t, x)}{\partial t} = -\frac{\partial}{\partial x_j}\left[a_j(t, x)\, w_{0k}(t, x)\right] - \frac{\partial}{\partial x_j}\Bigg[b_j(t, x) \sum_{m=1}^{k+1} \frac{1}{m!\,(2\pi i)^m} \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int \cdots \int F_m(x, y_1, p_1, \ldots, y_m, p_m) \times w_{m,k-m+1}(t, x, y_1, p_1, \ldots, y_m, p_m)\, dp_1 \cdots dp_m\, dy_1 \cdots dy_m\Bigg]$$

$$\sum_{n=1}^{k}\Bigg[p_n w_{k0}(t, x, y_1, p_1, \ldots, y_k, p_k) - \frac{\partial \delta(x - y_n)}{\partial x} \times b_j(t, y_n)\, w_{k-1,0}(t, y_n, y_1, p_1, \ldots, y_{n-1}, p_{n-1}, y_{n+1}, p_{n+1}, \ldots, y_k, p_k)\Bigg] = 0,$$

$$k = 0, 1$$

$$p_1 w_{11}(t, x, y, p) = \frac{\partial \delta(x - y)}{\partial x} b_j(t, y)\, w_{01}(t, y) \tag{14}$$

and also for terms $z_{00}(\tau, x)$ and $z_{01}(\tau, x)$ of the boundary layer part

$$\frac{\partial z_{00}(\tau, x)}{\partial \tau} = 0, \qquad \frac{\partial z_{01}(\tau, x)}{\partial \tau} = 0 \tag{15}$$

Under (14) we obtain for $w_{00}(t, x)$ the equation

$$\frac{\partial w_{00}(t, x)}{\partial t} = -\frac{\partial}{\partial x_j}\left[a_j(t, x)\, w_{00}(t, x)\right] - \frac{\partial}{\partial x_j}\Bigg[b_j(t, x)\frac{1}{2\pi i} \times \int_{-\infty}^{\infty} \int p^{-1} F_1(x, y, p) \frac{\partial \delta(x - y)}{\partial x_{j_1}} b_{j_1}(t, y)\, w_{00}(t, y)\, dp\, dy\Bigg]$$

It follows from the properties of the Laplace transform that

$$(2\pi i)^{-1} \int p^{-1} \exp(pt)\, dp = -\theta(-t) \tag{16}$$

because Re $p<0$. Hence, by definition of $F_1(x, y, p)$, we have

$$(2\pi i)^{-1}\int p^{-1}F_1(x, y, p)\,dp = -F_1(x, y, 0)$$

and therefore $w_{00}(t, x)$ satisfies the Fokker–Planck equation

$$\frac{\partial w_{00}(t,x)}{\partial t} = -\frac{\partial}{\partial x_j}[a_j(t,x)\,w_{00}(t,x)] + \frac{\partial}{\partial x_j}\left\{b_j(t,x)\frac{\partial}{\partial y_{j_1}} \times\left[b_{j_1}(t,y)\,F(x,y,0)\,w_{00}(t,y)\right]_{y=x}\right\}, \qquad j, j_1 = 1,\dots, N \tag{17}$$

For the second term $w_{01}(t, x)$ by (14) we have the equation

$$\begin{aligned}\frac{\partial w_{01}(t,x)}{\partial t} &= -\frac{\partial}{\partial x_j}[a_j(t,x)\,w_{01}(t,x)] - \frac{\partial}{\partial x_j}\Big[b_j(t,x)(2\pi i)^{-1}\\ &\quad\times\int_{-\infty}^{\infty}\int F_1(x,y,p)\,w_{11}(t,x,y,p)\,dp\,dy\\ &\quad+(2\pi i)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint F_2(x, y_1, p_1, y_2, p_2)\\ &\quad\times w_{20}(t,x,y_1,p_1,y_2,p_2)\,dp_1\,dp_2\,dy_1\,dy_2\Big]\end{aligned}$$

Under (14) we can express the function w_{11} through w_{01} and w_{20} through w_{00}. So, for $w_{01}(t, x)$ we obtain by (16) the equation

$$\begin{aligned}\frac{\partial w_{01}(t,x)}{\partial t} &= -\frac{\partial}{\partial x_j}[a_j(t,x)\,w_{01}(t,x)] + \frac{\partial}{\partial x_j}\Big\{b_j(t,x)\frac{\partial}{\partial y_{j_1}}\\ &\quad\times\left[b_{j_1}(t,y)\,F_1(x,y,0)\,w_{01}(t,y)\right]_{y=x}\Big\}\\ &\quad-\frac{\partial}{\partial x_j}\Big(b_j(t,x)\frac{\partial}{\partial y_{j_1}}\Big\{b_{j_1}(t,y)\frac{\partial}{\partial z_{j_2}}\\ &\quad\times[b_{j_2}(t,z)\,F_2(x,y,0,z,0)\,w_{00}(t,z)]_{z=y}\Big\}_{y=x}\Big)\end{aligned}$$

$$j, j_1, j_2 = 1,\dots, N \tag{18}$$

In order to give initial values for Eqs. (17) and (18), we must consider Eqs. (15). From these equations, by (13), we have $z_{00}(\tau, x) = 0$, $z_{01}(\tau, x) = 0$, and under the conditions (12) the initial values for Eqs. (17) and (18) have the form

$$w_{00}(0, x) = \rho(0, x), \qquad w_{01}(0, x) = 0 \tag{19}$$

Therefore the first two terms of the asymptotic expansion for $P(t, x)$ are determined from Eqs. (17) and (18) under initial values (19). The first two terms of the boundary layer part are zero.

Let us show that the third term of the boundary layer part is not zero and obtain a formula for it. It is easy to obtain under (10) the equations

$$\frac{\partial z_{02}(\tau, x)}{\partial \tau} = -\frac{\partial}{\partial x_j}[a_j(0, x)\, z_{00}(\tau, x)] - \frac{\partial}{\partial x_j}\left[b_j(0, x)\int_{-\infty}^{\infty}\int F_1(x, y, p) \times z_{10}(\tau, x, y, p)\, dp\, dy\right]$$

$$\frac{\partial z_{10}(\tau, x, y, p)}{\partial \tau} = p z_{10}(\tau, x, y, p)$$

By the conditions (12) we have

$$z_{10}(0, x, y, p) = -w_{10}(0, x, y, p) = -p^{-1}\frac{\partial \delta(x-y)}{\partial x_j} b_j(0, y)\, \rho(0, y)$$

$$z_{10}(\tau, x, y, p) = p^{-1} e^{pt} \frac{\partial \delta(x-y)}{\partial x_j} b_j(0, y)\, \rho(0, y)$$

$$\begin{aligned}\frac{\partial z_{02}(\tau, x)}{\partial \tau} &= \frac{\partial}{\partial x_j}\left[b_j(0, x)\int_{-\infty}^{\infty}\int F_1(x, y, p)\, p^{-1} e^{p\tau} \times b_{j_1}(0, y)\, \rho(0, y)\frac{\partial \delta(x-y)}{\partial x_j}\, dp\, dy\right] \\ &= \frac{\partial}{\partial x_j}\left\{b_j(0, x)\frac{\partial}{\partial y_{j_1}}\left[b_{j_1}(0, y)\, \rho(0, y)\int p^{-1} e^{p\tau} \times F_1(x, y, p)\, dp\right]_{y=x}\right\}\end{aligned}$$

Using the definition of $F_1(x, y, p)$ and (16), we get

$$\int p^{-1} e^{p\tau} F_1(x, y, p)\, dp = -\int_\tau^\infty K_1(s, x, y)\, ds$$

Consequently,

$$z_{02}(\tau, x) = z_{02}(0, x) - \frac{\partial}{\partial x_j}\left\{ b_j(0, x) \frac{\partial}{\partial y_{j_1}} \left[b_{j_1}(0, y)\, \rho(0, y) \times \int_0^\tau \int_{\tau_1}^\infty K_1(s, x, y)\, ds\, d\tau_1 \right]_{y=x} \right\}$$

and under (13)

$$z_{02}(0, x) = \frac{\partial}{\partial x_j}\left\{ b_j(0, x) \frac{\partial}{\partial y_{j_1}} \left[b_{j_1}(0, y)\, \rho(0, y) \times \int_0^\infty \int_{\tau_1}^\infty K_1(s, x, y)\, ds\, d\tau_1 \right]_{y=x} \right\}$$

Therefore for $z_{02}(\tau, x)$ we obtain the formula

$$z_{02}(\tau, x) = \frac{\partial}{\partial x_j}\left\{ b_j(0, x) \frac{\partial}{\partial y_{j_1}} \left[b_{j_1}(0, y)\, \rho(0, y) \times \int_\tau^\infty (s - \tau)\, K_1(s, x, y)\, ds \right]_{y=x} \right\}$$

We conclude this section with some remarks.

Remark 1. If $\xi(t, x)$ is a Gaussian, then for $P(x, t)$ we have the asymptotic expansion in powers of ε^2 because in the chain (10), only ε^2 is present. The non-Gaussianity of the perturbation in Eq. (1) is essential for the correction of the Fokker–Planck approximation (17) because under (18) and (19) $w_{01}(t, x) = 0$.

Remark 2. It follows from (17) and (18) that the second term of the asymptotic expansion for $P(t, x)$ can be expressed by the quadrature formula through the solution of the Fokker–Planck equation (17). One can convince oneself that this occurs for higher terms of the expansion. Note that there are many methods for the analysis of the Fokker–Planck equation.[19]

Remark 3. The boundary layer part, by (13), can be taken into consideration only for t near $t = 0$.

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